BEFORE THE CONTINUUM LIMIT

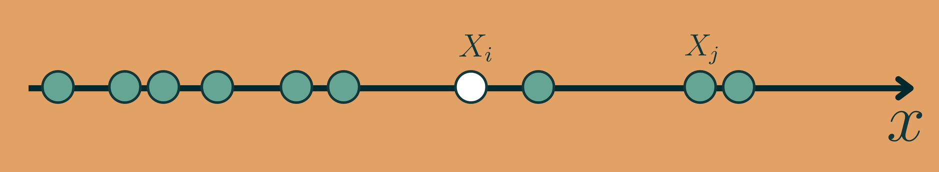
QUANTIFYING FLUCTUATIONS IN PARTICLE MODELS FOR COLLECTIVE MOTION

CONTEXT

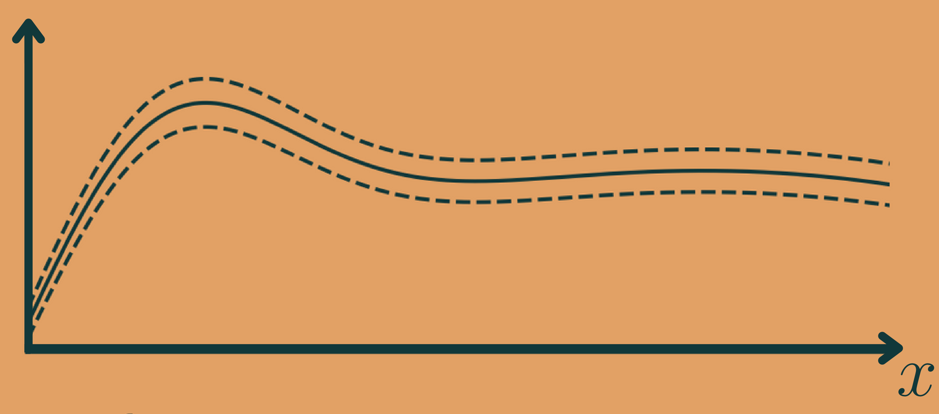
- When studying collective motion, we often consider an infinite number of individuals.
- In reality, particle number is **finite**
- How can we model large but finite numbers of individuals and retain stochastic effects?
- Finite particle effects can produce measureable patterns in the fluctuations

METHODS

1 Start from individual, random particles



End with smooth, stochastic density



2 Statistically, both formulations behave the same but the latter allows us to find exact results

We study N **Brownian** particles with positions: X_i
They interact with each other through a random and a deterministic coupling function

$$\mathrm{d}X_i = \frac{1}{N}\sum_{j=1}^{N} f(X_i - X_j)\mathrm{d}t + \frac{\sqrt{2D}}{N}\sum_{j=1}^{N} g(X_i - X_j)\mathrm{d}W_i(t),$$

1 We can write this as an equation for the stochastic density: $\rho(x,t) = \frac{1}{N}\sum_{n=1}^{N}\delta(x - X_n)$

$$\partial_t\rho = \underbrace{\partial_x\left(\rho(f*\rho)\right) + D\partial_x^2\left(\rho(g*\rho)^2\right)}_{\text{Deterministic contribution}} + \underbrace{\sqrt{\frac{2D}{N}}\partial_x\left(\sqrt{\rho}(g*\rho)\eta\right)}_{\text{Stochastic term}}$$

η: spatiotemporal white noise

2 By choosing suitable functions, F_k, we can determine analytic results for fluctuations, ξ_k, about the deterministic limit, ρ^*:

$$\langle\rho, F\rangle = \int_A \rho(x,t)F(x)\mathrm{d}x = \frac{1}{N}\sum_{n=1}^{N} F(X_n)$$

$$\xi_k(t) = \sqrt{N}(\langle\rho, F_k\rangle - \langle\rho^*, F_k\rangle)$$

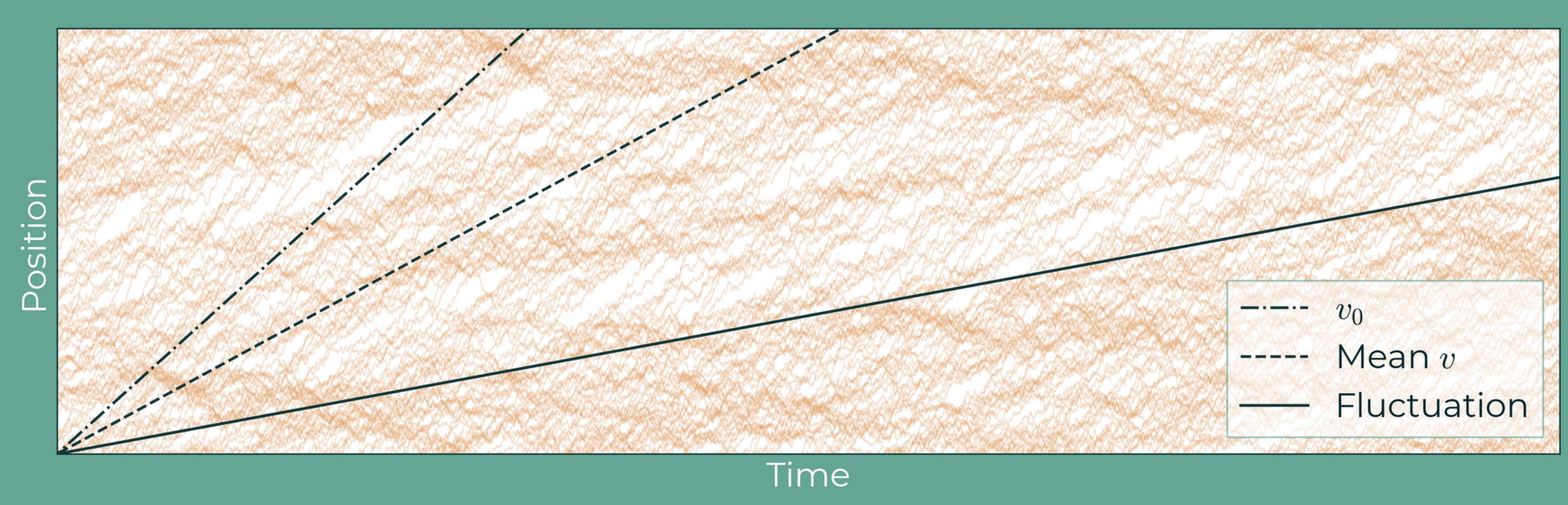
PHANTOM TRAFFIC JAMS

- Traffic jams travel slower than average vehicle velocity, moving backwards relative to the cars.
- Can use the general model to reproduce a continuum model of traffic
- Continuum model predicts stable flow of traffic
- Observe **stochastic waves** of density resembling phantom jams using **Fourier series:**

$$F_k = e^{-ikx}$$

$$\begin{aligned} f(x) &= v_0(1 - \delta(x)/\rho_{\text{jam}}) \\ g(x) &= 1 \end{aligned} \Rightarrow \partial_t\rho + \partial_x\left(\rho v_0(1 - \rho/\rho_{\text{jam}})\right) = D\partial_x^2\rho$$

max velocity (v_0) max density (ρ_{jam})



NONLINEAR DIFFUSION

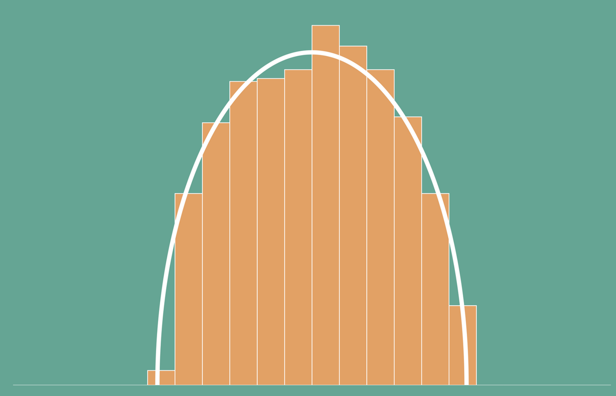
- Organisms do not always diffuse independently.
- We can model density dependent diffusion using the general model with a Dirac delta

$$g(x) = \delta(x) \Rightarrow \partial_t\rho = D\partial_x^2(\rho^3)$$

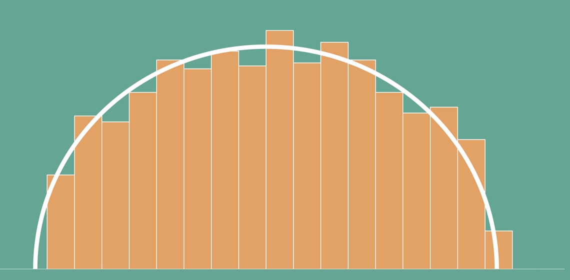
- Deterministic solution is an ellipse flattening over time.
- **How much do we expect the centre to move?**
- **How much will the general shape change?**
- Characterise the fluctuations about the **cumulative moments:**

$$F_k = x^k$$

- Find the variances in the **centre of mass** & **mean squared displacement**



Density flattens out as the particles diffuse



SWARMING OF ORGANISMS

- Consider organisms which try to catch up with others in front of them
- They diffuse more when they are very close to others to maintain reasonable distance.

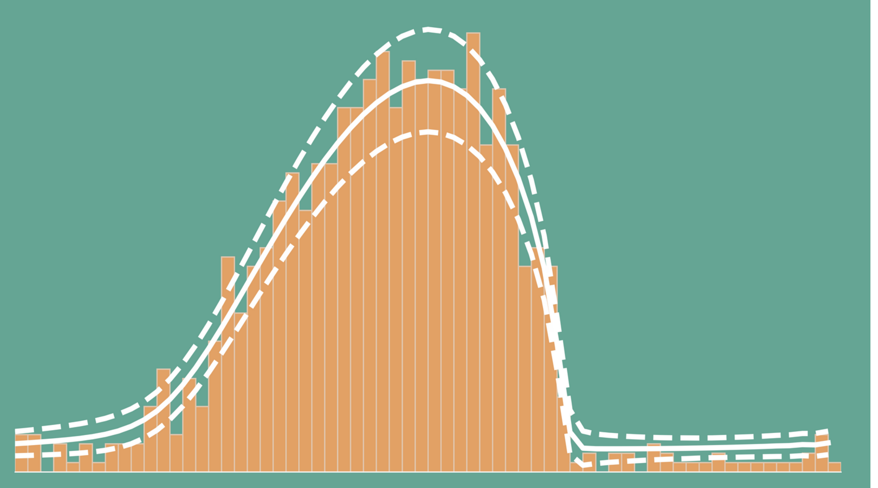
$$\begin{aligned} f(x) &= H(x)e^{-x} \\ g(x) &= \delta(x) \end{aligned}$$

$H(x)$ Heaviside function

- Solitary waves form but there is no closed form solution for the deterministic limit.
- Instead, Use a numerical solution of the discretised equation: ρ^*
- Study fluctuations in discretised elements, $k = 1, \dots, M$

$$F_k = \mathbf{1}_{\left[\frac{k-1}{M}, \frac{k}{M}\right]}$$

- Calculate the covariance of elements in this discretised version.



Swarm of organisms moving right with predicted variation in shape.

SYNCHRONISATION

- In the Kuramoto model: oscillators which try to align their phase.
- We added a non local coupling in space to a lattice of oscillators

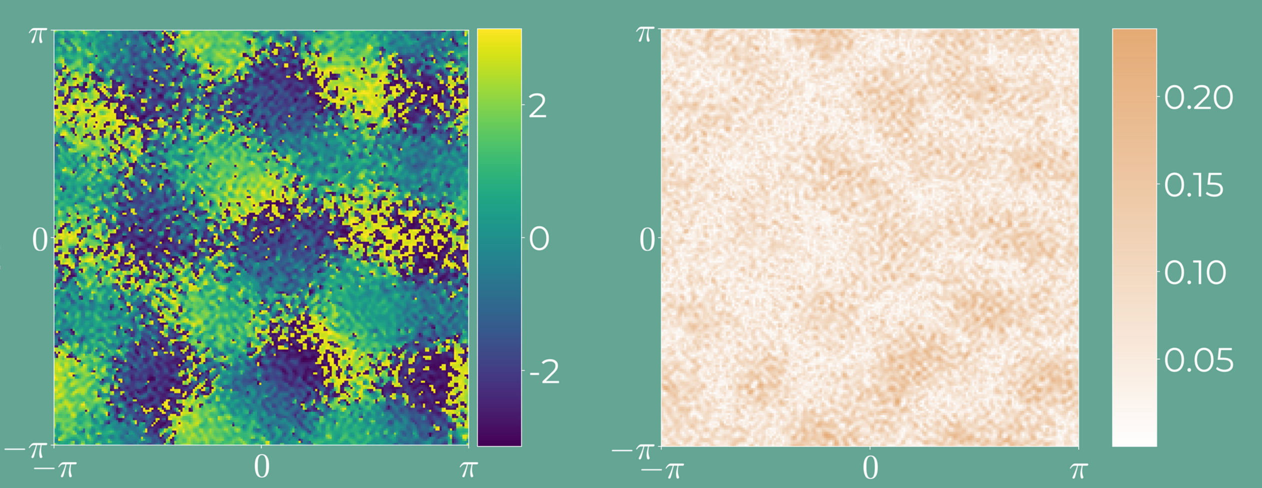
$$\mathrm{d}\vartheta_n = \frac{1}{N}\sum_{m=1}^{N} K(\vec{x}_n - \vec{x}_m)\sin(\vartheta_n - \vartheta_m)\mathrm{d}t + \sqrt{2D}\mathrm{d}W_n(t)$$

- **Continuum limit** predicts no patterns
- Stochastic patterns with spatial structure quantified by fluctuations in Fourier Series:

$$F_k = e^{-i(\vec{k}\cdot\vec{x}+\vartheta)}$$

Left: $\arg(\xi(\vec{x}))$ mean phase of oscillators at each lattice point

Right: $|\xi(\vec{x})|$ coherence



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